Application of the calculus of variations to the vertical cut off in cohesive frictionless soil

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E. Castillo and A. Luceño, University of Santander

The Author is to be congratulated on the introduction of the second variation in the analysis. In effect, the Legendre, Jacobi and Weierstrass conditions cannot be omitted in the search for an extremum. Luceño (1979) and Luceño & Castillo (1980a, b) introduced these conditions in the analysis of several existing variational methods in soil mechanics problems and demonstrated that Baker & Garber's and Chen's methods do not lead to extrema.

With respect to the Paper, the Author demonstrates that functional (21a) subject to (21b) and (21c) does not attain a minimum because, in an analysis of the second variation, the Jacobi condition is violated. This conclusion would have been better obtained by an analysis of the first variation which includes not only the Euler equation but also the transversality condition. This last condition is not satisfied by the solution given in the Paper, as it will be shown.

A straightforward check of the transversality condition leads to some problems because the auxiliary function H depends on α_F, but due to the fact that the Kötter conditions (32) and (33) must be satisfied by every potential slip line, the class of these lines can be initially reduced to those satisfying Kötter's conditions. With this assumption α_F and β_F become constants and the Euler equation remains unchanged. So the same class of extremals (25) is obtained.

The transverality condition now becomes (Bolza)

$$\bar{x}' H_{x'} + \bar{y}' H_{y'}\big|_F = 0 \qquad \text{(D1)}$$

where

$$\begin{aligned} \bar{x}(s) &= s \\ \bar{y}(s) &= 2h/3 \end{aligned} \qquad \text{(D2)}$$

are the equations of the free surface CD in parametric form.

Substitution of expressions (23) and (D2) in (D1) leads to

$$1 - g_2 y - \frac{2}{1+(y'/x')^2}\frac{y'}{x'}\left[(g_1 + g_2 x) - (1 - g_2 y)\frac{y'}{x'}\right]$$

$$+ 2\left[\beta_F + \arctan g\left(\frac{y'}{x'}\right)\right](g_1 + g_2 x)\big|_F = 0 \qquad \text{(D3)}$$

Taking into account (25a) and introducing (34) in the left-hand side of (D3) one gets a value of 1·865 which is obviously not zero. In consequence, the first variation does not vanish because the transversality condition is violated.

This fact demonstrates that Kötter's conditions are not compatible with the transversality condition in the point F.